W5/2 Finitely generated modules over PIDs

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Cor 3.4 Let A be a PID. If $F \in A\text{-Mod}$ is f.g. free of rank n, and $M \leq F$, there exist a basis $e_1, \dots, e_n$ of F, $m \leq n$, $d_1, \dots, d_m \in A$ such that M is free w. basis $d_1 e_1, \dots, d_m e_m$ and $d_1 \mid d_2 \mid \dots \mid d_m$.

Proof: Let $f_1, \dots, f_n \in F$ be a basis. M is f.g. by Thm 3.1.

Let $M = \langle g_1, \dots, g_k \rangle_A$. Let $g_i = \sum_{j=1}^{n} c_{ij} f_j$ $(c_{ij} \in A)$.

$$\Rightarrow \underbrace{\begin{pmatrix} g_1 \\ \vdots \\ g_k \end{pmatrix}}_{=:\underline{g}} = C \underbrace{\begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}}_{=:\underline{f}} \quad \text{with } C = (c_{ij})_{i,j} \in A^{k \times n}$$

By Thm 3.2, $\exists U \in GL_k(A)$, $V \in GL_n(A)$ s.t.

$$UCV = \begin{pmatrix} d_1 & & & & & 0 \\ & d_2 & & & & \\ & & \ddots & & & \\ & & & d_m & & \\ & & & & 0 & \\ 0 & & & & & \ddots \; 0 \end{pmatrix} =: D$$

for some $m \geq 0$, $d_1 \mid d_2 \mid \dots \mid d_m$ in A, $d_i \neq 0$

Let $V^{-1} = (v_{ij})$, and define $e_i := \sum_{j=1}^{n} v_{ij} f_j$. Then $e_1, \dots, e_n$ is a basis of F.

$$\Rightarrow \underline{g} = U^{-1} D V^{-1} \underline{f} = U^{-1} \begin{pmatrix} d_1 e_1 \\ \vdots \\ d_m e_m \\ 0 \\ \vdots \\ 0 \end{pmatrix} \Rightarrow d_1 e_1, \dots, d_m e_m \text{ generate } M$$

& are lin. independent $\square$

Thm 3.5 (Structure Theorem for f.g. modules over a PID).

Let A be a PID, M f.g. A-module.

(I) $M \cong A^r \oplus A/(d_1) \oplus \dots \oplus A/(d_s)$ with $r \geq 0$, and proper, nonzero ideals $(d_1) \supseteq (d_2) \supseteq \dots \supseteq (d_s)$. The constants r, s and the ideals (d_i) are uniquely determined by M. (The (d_i) are the *invariant factors*)

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prime elements $p_1, \ldots, p_k$, $1 \leq e_{i,1} \leq e_{i,2} \leq \ldots \leq e_{i,t_i}$ and this representation is unique.

Existence Proof:

Existence of I: Let $F \xrightarrow{\pi} M \to 0$ be an epi with F f.g. free.

$\overset{\text{Cor 3.4}}{\Longrightarrow} \exists$ basis $e_1, \ldots, e_n$ of F s.t. $d_1 e_1, \ldots, d_s e_s$ is a basis of $\mathrm{Ker}(\pi)$

$$\Rightarrow M \cong F/\mathrm{Ker}(\pi) \cong \bigoplus_{i=1}^{s} \underbrace{Ae_i / Ad_i e_i}_{\cong A/(d_i)} \oplus \bigoplus_{i=s+1}^{n} \underbrace{Ae_i}_{\cong A} \cong A^{n-s} \oplus A/(d_1) \oplus \ldots \oplus A/(d_s).$$

Existence of (II): Let $0 \neq d \in A \setminus A^\times$.

$\Rightarrow d = p_1^{e_1} \cdots p_k^{e_k}$ with pw. non-associated prime elts. $p_1, \ldots, p_k$, $e_i \geq 1$

p_i, p_j coprime for $i \neq j$ $\Rightarrow (p_i^{e_i}) + (p_j^{e_j}) = A$.

$$\Longrightarrow A/(d) = A/(p_1^{e_1}) \cdots (p_k^{e_k}) \overset{\text{C.R.T., T1.3}}{\cong} A/(p_1^{e_1}) \oplus \ldots \oplus A/(p_k^{e_k}).$$

Apply to (I) & gather the terms as in (II). $\square$ (Existence)

Lemma 3.6 Let B be a ring, $I, J \trianglelefteq B$

(1) $B/I \otimes_B B/J \cong B/(I+J)$ (via $b_1 \otimes b_2 \mapsto b_1 b_2$)

(2) If B is a domain, $I \neq 0$, then $\mathcal{F}(B) \otimes_B B/I = \underline{0}$.

Proof: (1) $B/I \times B/J \longrightarrow B/(I+J)$, $(b_1 + I, b_2 + J) \mapsto b_1 b_2 + I + J$

is well-defined & bilinear, induces $\varphi: B/I \otimes_B B/J \longrightarrow B/(I+J)$.

Let $\psi_0: B \to B/I \otimes_B B/J$, $b \mapsto b \otimes 1 = 1 \otimes b$. Since $I \subseteq \mathrm{Ker}(\psi_0)$, $J \subseteq \mathrm{Ker}(\psi_0)$,

also $I + J \subseteq \mathrm{Ker}(\psi_0)$ ... $\psi: B/(I+J) \to B/I \otimes B/J$

Let $\psi_0 : B \to B/I \otimes_B B/J$, $b \mapsto b \otimes 1 = 1 \otimes b$. Since $I \subseteq \mathrm{Ker}(\psi_0)$, $J \subseteq \mathrm{Ker}(\psi_0)$, also $I + J \subseteq \mathrm{Ker}(\psi_0) \Rightarrow$ induces $\psi : B/(I+J) \to B/I \otimes B/J$.

φ, ψ are inverse to each other.

(2) Let $K := \mathcal{F}(B)$. Then $K \otimes_B B/I \cong KB/KI = K/K = 0$. $\square$

<u>Lemma 3.7</u> If A is a PID, every nonzero prime ideal is maximal

<u>Proof:</u> Let $0 \neq P \in \mathrm{Spec}(A) \Rightarrow P = (p)$ with $p \in A$ prime element.

Let $a \notin P$. <u>Show:</u> $(a, p) = A$.

$\exists b \in A \setminus P$: $(a,p) = (b) \Rightarrow p = bc$ with $c \in A \overset{p \nmid b}{\Rightarrow} p \mid c \Rightarrow c = pd$ with $d \in A$

$\Rightarrow \cancel{p} = \cancel{p} bd \Rightarrow bd = 1 \Rightarrow b \in A^\times \Rightarrow (b) = A$. $\square$

<u>Uniqueness Proof of Thm 3.5:</u>

<u>Uniqueness of II:</u> Let $K = \mathcal{F}(A) \overset{L3.6}{\Rightarrow} K \otimes_A M \cong K^r$, so r is unique.

$$\left(A/(p_i)^{e_{i,t_i}}\right) \otimes_A M \underset{L3.6,}{\cong} \bigoplus_{j=1}^{t_i} A/(p_i)^{e_{i,j}} \oplus \left(A/(p_i)^{e_{i,t_i}}\right)^r$$

$(p_i)^{e_{i,t_i}} + (p_i)^{e_{i,j}} = (p_i)^{e_{i,j}}$ since $e_{i,j} \leq e_{i,t_i}$

for $i \neq v$: $(p_i)^{e_{i,t_i}} + (p_v)^{e_{v,j}} = A$ since $p_i \neq p_v$.

<u>So suffices:</u> If $M \cong A/(p)^{e_1} \oplus \dots \oplus A/(p)^{e_t}$ with $p \in A$ prime element and $1 \leq e_1 \leq \dots \leq e_t$, then the e_i are unique.

By L3.7, $L := A/(p)$ is a field. Consider

$$M \supseteq pM \supseteq p^2 M \supseteq \dots \supseteq p^{e_t} M$$

$\Rightarrow p^i M / p^{i+1} M$ is an L-vectorspace, $\dim_L \left(p^i M / p^{i+1} M\right) = \left|\{ j : e_j \geq i \}\right|$

$$\left[p^i \left(A/(p)^{e_j}\right) = (p^i, p^{e_j})/(p^{e_j}) = \begin{cases} (p^i)/(p^{e_j}) = p^i A / p^{e_j} A & \text{if } i \leq e_j \\ (p^{e_j})/(p^{e_j}) = 0 & \text{if } i \geq e_j \end{cases} \text{ and } (p^i)/(p^{i+1}) \cong A/(p) \right]$$

$\Big[\; p^i\left(\cdot/(p)^{e_j}\right) = \cdots/(p^{e_j}) = \begin{cases} \cdots(p^{\cdot}) & p^{\cdot}A \\ (p^{e_j})/(p^{e_j}) = 0 & \text{if } \; i \geqslant e_j \end{cases}$ and $\cdot/(p^{i+1}) = \cdot/(p) \;\Big]$

But knowing how many $e_j \geqslant i$ for each i uniquely determines the e_j !

<u>Uniqueness of (I)</u>: Let $p_1, \dots, p_k$ be pw. non-associated prime elements and $e_{i,j} \geq 0$ s.t.

$$d_j \simeq p_1^{e_{1,j}} \cdot \ldots \cdot p_k^{e_{k,j}} \quad \forall j$$

$$\xrightarrow{d_j \mid d_{j+1}} \forall i, j: \quad e_{i,j} \leqslant e_{i,j+1}$$

By (II), the exponents $e_{i,j}$ are uniquely determined by M. This determines the d_j up to associativity. $\square$